Harnessing SMT solvers for TLA$^+$ Proofs

Stephan Merz and Hernán Vanzetto

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Introduction

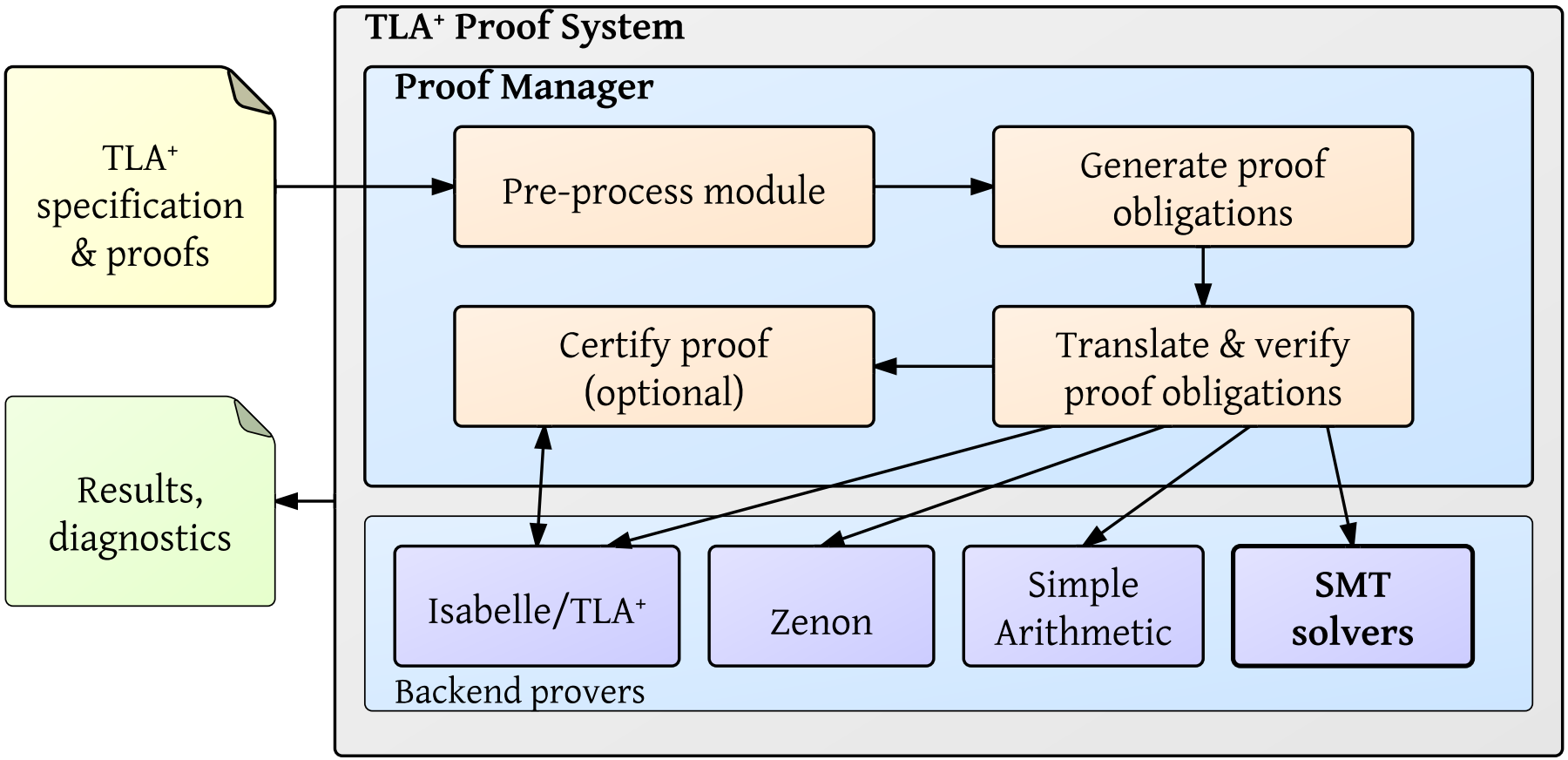
TLA$^+$ proof language:

- Hierarchical proof structure
- Top-down development: users refine assertions until they are "obvious"
- Leaf steps verified by automatic backend provers
  - invoke proof method
  - cite necessary assumptions and facts
  - expand definitions

TLA$^+$ Proof System:

- Mechanically checks TLA$^+$ proofs
- Currently proves only non-temporal fragment
- Supported by the TLA$^+$ Toolbox

Architecture of TLAPS



Current backend provers

- Isabelle/TLA$^+$
  - Faithful encoding of TLA$^+$ over Isabelle's meta-logic
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- SMT
  - Available since the last public version of TLAPS (v1.0)
  - Based on type inference

Motivation

Typical proof obligations usually contain a mix of arithmetic, sets, functions, which the older backends were not able to handle at once

SMT solvers offer a combination of:

+ First-order reasoning
+ Decision procedures for other theories ($=$, linear arithmetic, ...)

SMT input languages:

- Based on many-sorted first-order logic
- Predefined Bool and integer sorts
- Uninterpreted functions, if-then-else function

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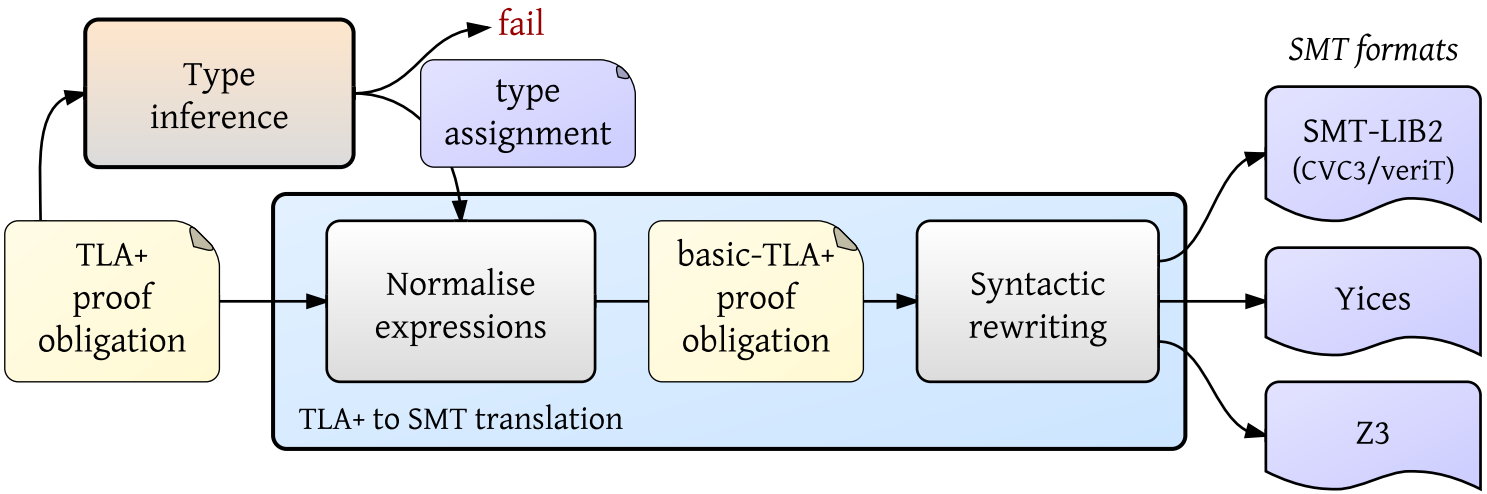
First approach: a backend based on type inference



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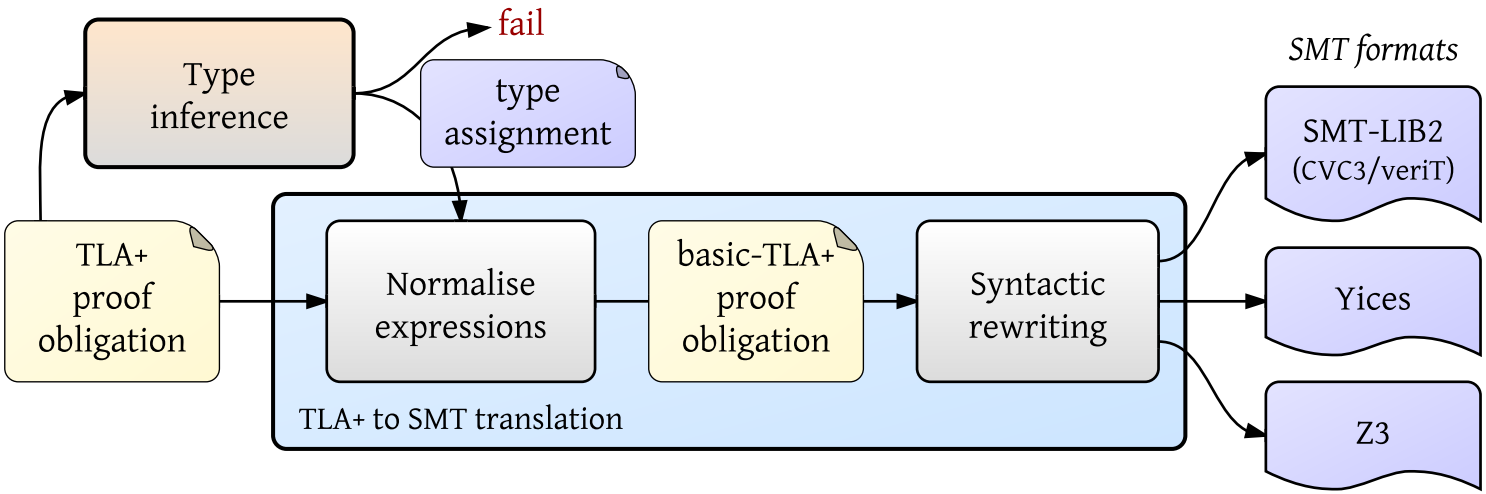

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  - Ad-hoc type system for TLA$^+$ terms

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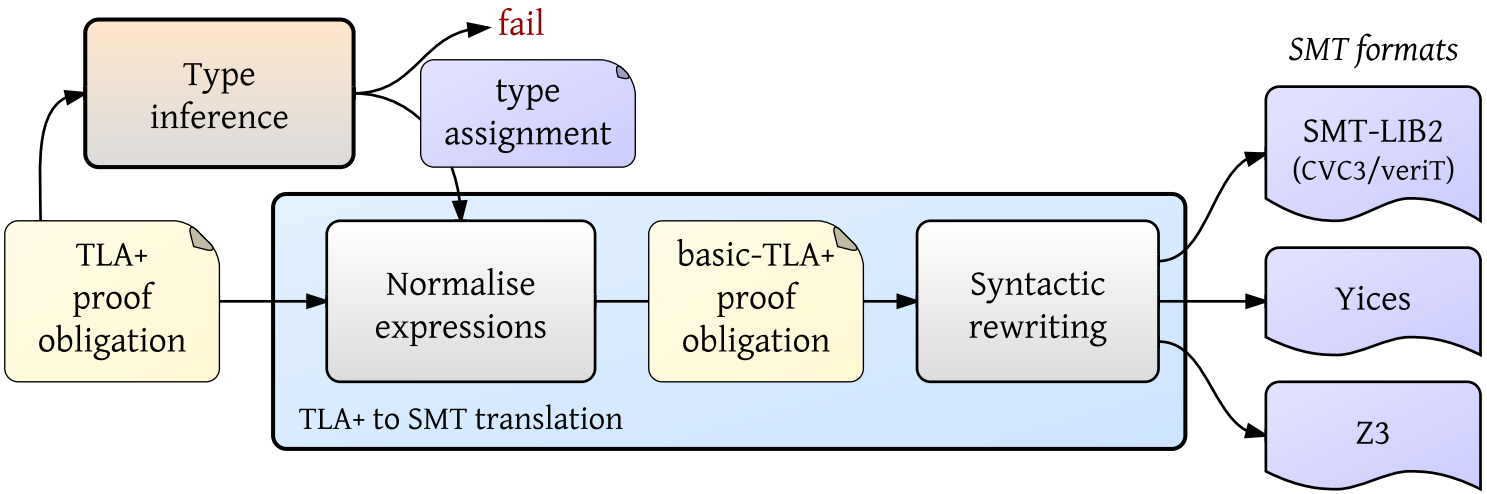

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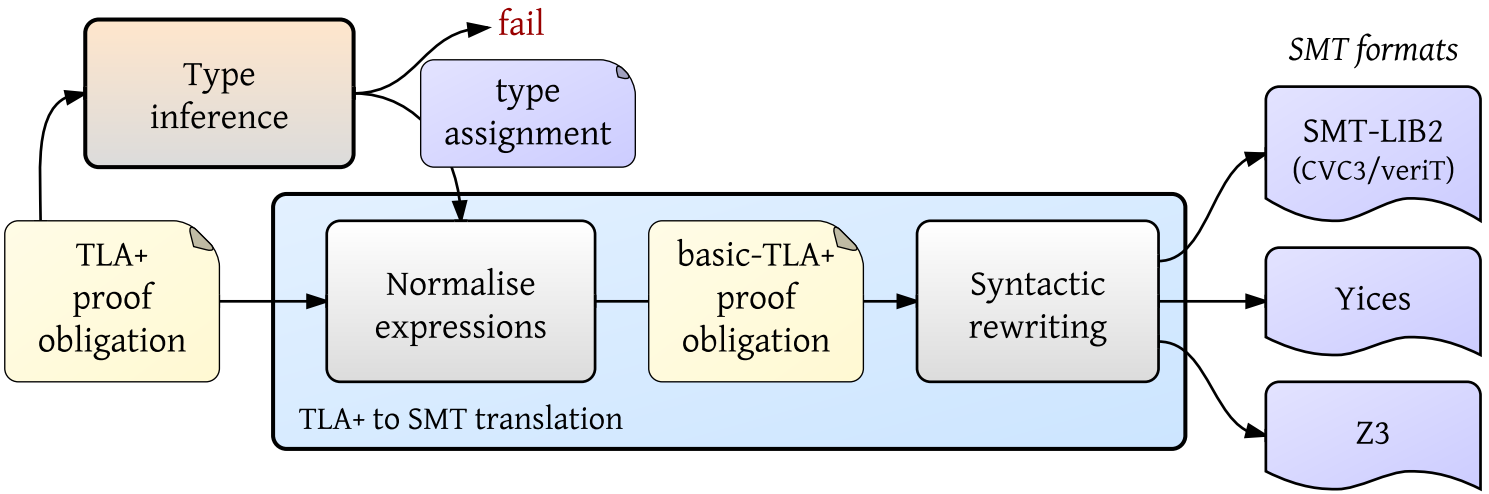

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- Soundness: incorrect typing can make invalid theorems provable
  - $x \notin \mathit{Int} \Rightarrow x + 0 = x \qquad ; \qquad (\neg\neg X) = X$
- **Safe types**: $\perp$, set($\perp$), set(set($\perp$)), ...
- **Typing hypotheses** are available facts of the form
  $x \approx \mathit{exp}$ and $\forall \vec{y} \in \vec{S} : f(\vec{y}) \approx \mathit{exp}$ with $\approx \in \{=, \in, \subseteq\}$

First approach: a backend based on type inference

Well-typed TLA$^+$ formulas are translated to SMT input formats

- **Basic TLA$^+$ expressions** contain only operators that have a direct representation in SMT formats (logical, arithm. and IFs)
- Sets, functions, records, tuples encoded as uninterpreted functions

Example

$$x :: \mathbb{Z} \quad \vdash \quad x \in \mathit{Int} \Rightarrow x + 0 = x \quad \longrightarrow \quad x + 0 = x$$

$$a :: \bot \ ; \ S, T :: \mathit{set}(\bot) \vdash \quad a \in S \cup T \quad \longrightarrow \quad S(a) \vee T(a)$$

Type information for variables usually provided by **type invariants**

Toy example

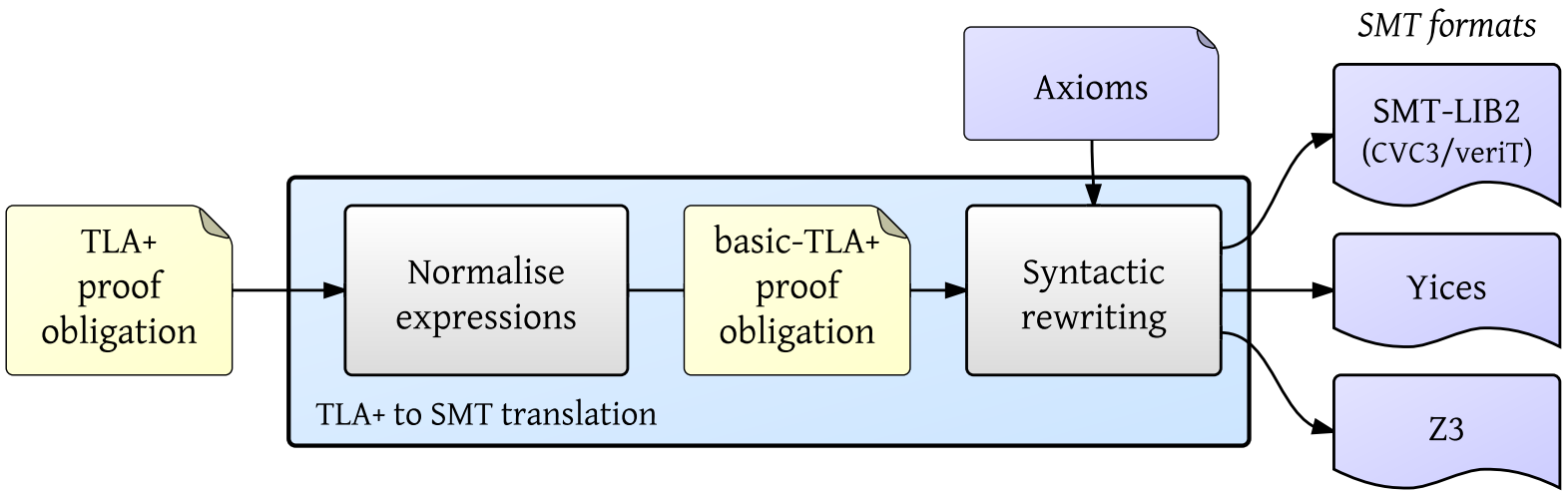
$$\begin{array}{lll} \textsc{axiom}\ \mathit{NatInduction} \equiv & \textsc{assume} & \textsc{new}\ \ P(\_), \\ & & P(0), \\ & & \forall n \in \mathit{Nat} : P(n) \Rightarrow P(n+1) \\ & \textsc{prove} & \forall n \in \mathit{Nat} : P(n) \end{array}$$

Toy example

AXIOM $NatInduction \equiv$ ASSUME NEW $P(\_)$,
$P(0)$,
$\forall n \in Nat : P(n) \Rightarrow P(n+1)$
PROVE $\forall n \in Nat : P(n)$

THEOREM $GeneralNatInduction \equiv$
ASSUME NEW $P(\_)$,
$\forall n \in Nat : P(n) \in$ BOOLEAN, *(typing hypothesis)*
$\forall n \in Nat : (\forall m \in 0..(n-1) : P(m)) \Rightarrow P(n)$
PROVE $\forall n \in Nat : P(n)$
$\langle 1 \rangle$. DEFINE $Q(n) \equiv \forall m \in 0..n : P(m)$
$\langle 1 \rangle$1. $Q(0)$ BY *SMT*
$\langle 1 \rangle$2. $\forall n \in Nat : Q(n) \Rightarrow Q(n+1)$ BY *SMT*
$\langle 1 \rangle$3. $\forall n \in Nat : Q(n)$ BY $\langle 1 \rangle$1, $\langle 1 \rangle$2, *NatInduction*,*SMT*
$\langle 1 \rangle$4. QED BY $\langle 1 \rangle$3, *SMT*

Second approach: untyped encoding



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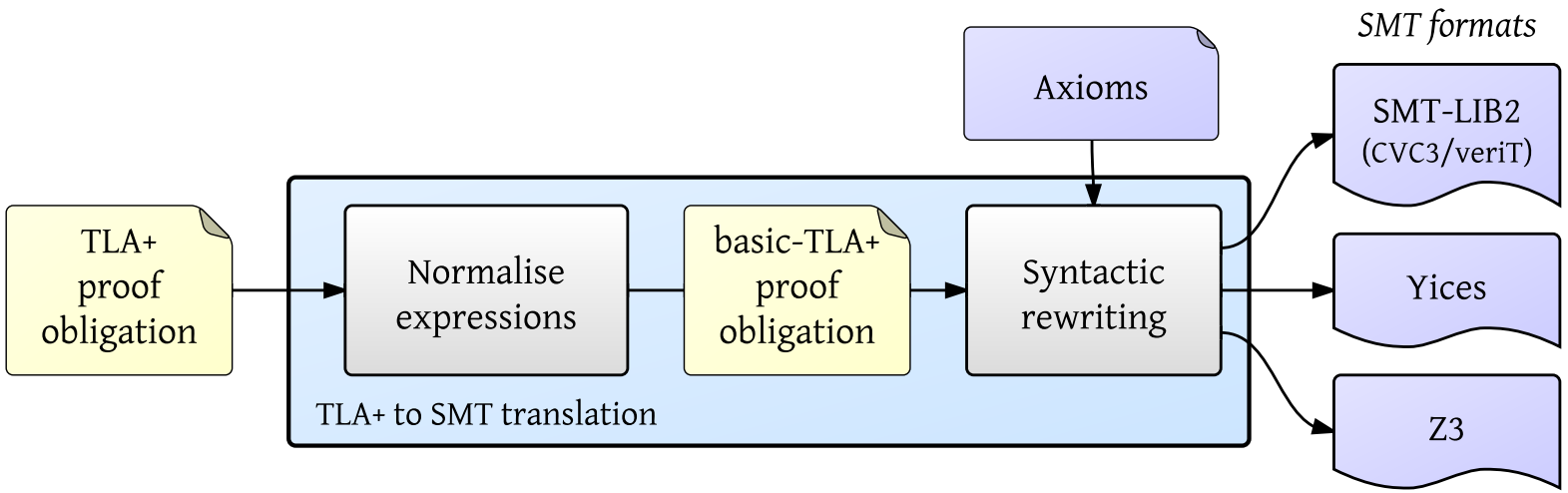

- TLA$^+$ terms are mapped to a **unique SMT sort** U
- Operators are uninterpreted functions or predicates
  - $\texttt{union} : U \times U \to U$ $\quad$ $\texttt{in} : U \times U \to \mathit{Bool}$
- Operators' semantics are defined **axiomatically**
  - Axiom for $\cup$: $\quad \forall x, S, T : U.\ \ (x \in S \cup T) = (x \in S \vee x \in T)$
  - Primitive operators ($\in$, $f[x]$, DOMAIN) are left uninterpreted
- Functions are related to its argument by $\texttt{apply} : U \times U \to U$

Encoding arithmetic

- Arithmetic expressions are **lifted** to elements on sort U
- Embedding function $\phi : \mathit{Int} \to U$ (uninterpreted and injective)
- 42 is encoded as $\phi(42)$
  $x \in \mathit{Int}$ is encoded as $\exists n : \mathit{Int}.\ x = \phi(n)$

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- Arithmetic operators are homomorphically embedded using ϕ

$$+_U : U \times U \to U$$

Axiom for $+$: $\forall m, n : \mathit{Int}.\ \ \phi(m)\ +_U\ \phi(n)\ =\ \phi(m+n)$

Example

$$\forall x \in \mathit{Int} : x + 0 = x$$

$$\longrightarrow\ \ \forall x : U.\ (\exists n : \mathit{Int}.\ x = \phi(n))\ \Rightarrow\ x +_U \phi(0) = x$$

Normalisation: removing non-basic operators

1. Grounding expressions: rewrite based on operator semantics
   - $[\![x \in e]\!] \equiv [\![x]\!] \in [\![e]\!]$ $\quad$ $[\![e_1 \vee e_2]\!] \equiv [\![e_1]\!] \vee [\![e_2]\!]$
   - $[\![x \in e_1 \cup e_2]\!] \equiv [\![x \in e_1 \vee x \in e_2]\!]$
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2. Disambiguation of equalities by inferred kinds
   - $[\![S = T]\!] \equiv \forall x : [\![x \in S \Leftrightarrow x \in T]\!]$ (when S, T are sets)
   - $S = \{a\} \cup \{\} \longrightarrow \forall x : x \in S \Leftrightarrow x = a \vee \text{FALSE}$

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3. Term-rewriting of top-level equalities
   - ASSUME $T = \{1,2\} \longrightarrow \forall x : (x = 1 \vee x = 2) \Rightarrow x \in Int$
     PROVE $T \subseteq Int$

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4. Abstraction of non-basic operators
   - $\forall a : P(\{a\} \cup \{\}) \Leftrightarrow P(\{a\}) \longrightarrow \forall a, s_1, s_2 : \wedge\ s_1 = \{a\} \cup \{\}$
     $\wedge\ s_2 = \{a\}$
     $\Rightarrow P(s_1) \Leftrightarrow P(s_2)$

Experimental results

- *N*-process Bakery algorithm
  - includes some basic arithmetic
- Memoir security architecture
  - mostly based on records
- Module Cardinality of finite sets

	Original		Typed-SMT/Z3		Untyped-SMT/Z3	
	size	time	size	time	size	time
Bakery	120	15.66	3	2.76	4	0.67
Memoir	424	7.31	14	5.08	14	1.11
Cardinality	185	2.12	-	-	54	0.88

(length = number of non-trivial proof-obligations ; time in seconds)

- Original = proof using Zenon, Isabelle/TLA$^+$, SimpleArithmetic

Conclusions

	Typed encoding	Untyped encoding
Handled fragment	first-order logic, sets, functions, records, tuples	
	☹ no sets of sets	☺ CHOOSE operator
Efficiency	☺ scales well for large formulas	☹ more quantifiers
Type inference	☹ may fail for valid obligations	☺ delegated to the solver
	☹ may require logically unnecessary typing hypotheses	☺ no need of typing hypotheses ; preferred by users
Soundness analysis	☹ non-trivial ; relies on type inference	☺ immediate ; all axioms are theorems

Future work

Work in progress: Merge both encodings

- Based on the untyped encoding
- Instantiate arithmetic expressions using type information

Future work:

- Adapt this translation to use ATPs with arithmetic (Spass+LA)
- Interpret the solvers output and certify it with Isabelle/TLA$^+$

Example: how the SMT solver use the axioms

Consider the TLA$^+$ proof obligation

$$\forall x \in \mathit{Int} : x + 0 = x$$

which is translated as

$$\forall x : U.\ (\exists n : \mathit{Int}.\ x = \phi(n)) \ \Rightarrow\ x +_U \phi(0) = x.$$

By Skolemization, the solver introduces a new constant, say n, of sort Int, such that $x = \phi(n)$. It can then reason as follows:

$$\begin{aligned}
x \ +_U \ \phi(0) &= \phi(n) \ +_U \ \phi(0) && (x = \phi(n)) \\
&= \phi(n + 0) && (\text{by axiom of } +_U) \\
&= \phi(n) && (\text{by the SMT arithmetic procedure}) \\
&= x && (x = \phi(n))
\end{aligned}$$

Encoding of CHOOSE

1. Any expression $\text{CHOOSE}\ \ x : P(x)$ can be abstracted to a new variable s, for which the following equality is asserted:

$$s = \text{CHOOSE}\ \ x : P(x)$$

2. Use of the following TLA$^+$ theorem to ground the expression

$$y = (\text{CHOOSE}\ \ x : P(x)) \ \Rightarrow\ \big((\exists x : P(x)) \Rightarrow P(y)\big)$$

3. Determinacy of CHOOSE . For every pair of expressions $\text{CHOOSE}\ \ x : P(x)$ and $\text{CHOOSE}\ \ x : Q(x)$ that appear in the proof obligation, we add the following axiom:

$$(\forall x : P(x) \Leftrightarrow Q(x)) \Rightarrow (\text{CHOOSE}\ \ x : P(x)) = (\text{CHOOSE}\ \ x : Q(x))$$